# Coupled Bending-Torsional Dynamic Behavior of a Cantilever Beam Carrying Multiple Point Masses

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***Abstract*—In this study, we examine the coupled bending-torsional dynamic behavior of a cantilever beam carrying point masses along the span. The eigenfrequencies were found using the Extended Galerkin Method (EGM) and validated by the finite element analysis software ANSYS®. Mainly, two cases were investigated: (i) a beam carrying a moveable mass along the span and (ii) a beam carrying two masses (one stationary tip mass and one moving along the span). Free vibrational analysis was carried out to demonstrate the effect of the external masses and their location on the natural frequencies. The results were in perfect agreement with the finite element analysis. A coupled bending-torsional behavior is ensured using a kite-type beam cross section. A validation of the methodology to the literature is also present for the uncoupled dynamic behavior. Overall, the results presented in this paper indicate that the dynamical behavior of beams is highly dependent on the location and magnitude of the external mass. A natural frequency decrease was observed as the mass approached the tip of the beam. Moreover, the addition of a tip mass to such system was helpful for dynamic stability. From a practical point of view, this study will be useful in the design of engineering structures that carry external stores.**

***Index Terms*—external store, bending torsion coupled dynamics, Extended Galerkin Method, ANSYS**

## I. Introduction

Both in commercial and in military aviation, wings are designed to carry external stores, such as engines, missiles, weapons, other defense equipment, or a combination of them all. Today's technology requires flexibility in the removal and replacement of these stores, where bombs, rockets, and engines are designed to be separable from the wing and reattachable in various combinations, which should be investigated in different aspects of flight.

From the dynamical viewpoint, both the location and the weight of an external store on an aircraft's wing have profound effects on the behavior of the structure; therefore, a careful inspection is required to determine how these parameters affect the dynamical response of the structure.

So far, several studies have been conducted to investigate the effect of external stores on the dynamical characteristics of beam structures with various boundary conditions. Laura, Pombo, and Susemihl [1] and Parnell and Cobble [2] determined the natural frequencies of a fixed-free uniform beam carrying a concentrated tip mass by solving the transcendental frequency equations numerically. Gurgoze [3] presented an approximate formulation to evaluate first two natural frequencies and mode shapes of beams with an arbitrary number of concentrated masses. Gokdag and Kopmaz [4] investigated the coupled bending and torsional vibration of a beam with a linear spring grounded at the tip mass and a torsional spring connected at the end. The effect of the external stores on the dynamical characteristics of thin-walled composite beams was presented by Na and Librescu [5] and Librescu and Ohseop [6].

In addition to the dynamic behavior, the aeroelastic characteristics are significantly affected by these external stores. Fazelzadeh, Mazidi, and Kalantari [7, 8], Runyan and Watkins [9], and Xu and Ma [10, 11] studied the effect on the flutter characteristics of a uniform beam

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with an external store. Gern and Librescu [12] investigated the effect of externally mounted stores on the thin-walled composite wing aeroelastic behavior. Wang, Wan, Wu, and Yang [13] modeled and solved the wing/engine system using the finite element method (FEM) to capture aeroelastic instabilities. Kashani, Jayasinghe, and Hashemi [14] solved the dynamic behavior of a coupled bending-torsional beam that is under a combined axial end moment load using the dynamic FEM.

Understanding the dynamical behavior of beams with multiple point masses carries a great deal of importance for the design of engineering structures, yet the number of studies that have been devoted to this area is very limited. Within this context, in this study, we focus on the coupled bending-torsional vibrational analysis of a cantilever beam carrying multiple point masses along the span. The eigenfrequencies were found using both the Extended Galerkin Method (EGM) and the finite element software ANSYS®. Three cases were analyzed: (i) a beam with a square cross section carrying a tip mass, (ii) a beam with a kite-type cross section carrying a mass at different locations along the span, and (iii) a beam with a kite-type cross section carrying two masses (one stationary and located at the tip and the other placed at various locations along the beam span). Overall, the results presented in this study will be helpful for understanding the dynamical behavior of beams carrying external masses. In addition, from a practical point of view, these results will be utilized in the design of aeronautical structures, such as an aircraft wing carrying both missiles and engines.

## II. Governing Equations

Considering a fixed-free beam carrying a point mass of $M_S$ at $x = x_S$, given as in Fig. 1, the equations of motion featuring bending displacement and angle of twist can be represented as follows [7]:

$$m\ddot{w} + me\ddot{\theta} + EIw'''' + M_S\delta_D(x - x_S)\ddot{w} = 0,$$
$$mk_m^2\ddot{\theta} + me\ddot{w} - GJ\theta'' + I_S\delta_D(x - x_S)\ddot{\theta} = 0, \quad (1)$$

where $w$ and $\theta$ are the bending displacement and angle of twist of the beam, respectively; $m$, $EI$, and $GJ$ are the mass per unit length, bending rigidity, and torsional rigidity of the beam, respectively; and $k_M$ is the mass radius of gyration. $e$ is defined by the distance between the shear center and the center of gravity of the beam's cross section, noting that when $e = 0$, decoupling of bending and twist motion occurs.

The mass of the store and store mass moment of inertia are represented by $M_S$ and $I_S$, respectively. Here $x_S$ is the coordinate of the location of the external mass. In order to locate the mass at the span, the delta Dirac function, $\delta_D(x)$, is used, as suggested by Fazelzadeh, Mazidi, and Kalantari [7].

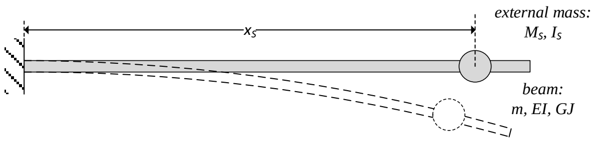


Figure 1. Cantilever beam with a point mass.

The displacement of the aforementioned governing equation is discretized as follows, where $w(x,t)$ is the bending displacement and $\theta(x,t)$ is the twist angle:

$$w(x,t) = N_w(x)^T q_w(t),$$
$$\theta(x,t) = N_\theta(x)^T q_\theta(t). \quad (2)$$

Here, $N_w(x)$ and $N_\theta(x)$ are the shape functions and $q_w(t)$ and $q_\theta(t)$ are the generalized coordinate vectors, as given below. EGM is applied to solve the natural frequencies of the beam configurations. The main advantage of the EGM is its ability to solve the problem by selecting the weighting functions by only fulfilling the geometric boundary conditions. This method does not require natural boundary conditions to be satisfied; hence, it provides significant advantages for structures with complex boundary conditions [15].

The shape functions of the governing equations are chosen to be polynomials to satisfy only the geometric boundary conditions as follows:

$$N_w(x)^T = [x^2 \ x^3 \ \dots \ x^n],$$
$$N_\theta(x)^T = [x \ x^2 \ \dots \ x^n]. \quad (3)$$

Here, $[\ ]^T$ represents the transpose and $n$ is the degree of polynomial. In order to achieve the desired accuracy, the polynomial degree is chosen to be $n = 9$.

After obtaining the weak Galerkin form, the governing equations of motion take the following form:

$$m\int_0^L \delta q_w{}^T N_w N_w{}^T \ddot{q}_w dx + me\int_0^L \delta q_w{}^T N_w N_\theta{}^T \ddot{q}_\theta dx$$
$$+ M_s[\delta q_w{}^T N_w N_w{}^T \ddot{q}_w]_{x=x_S} + EI\int_0^L \delta q_w{}^T N''_w N''^T_w q_w dx = 0$$
$$mk_m{}^2\int_0^L \delta q_\theta{}^T N_\theta N_\theta{}^T \ddot{q}_\theta dx + me\int_0^L \delta q_\theta{}^T N_\theta N_w{}^T \ddot{q}_w dx \quad (4)$$
$$+ I_s[\delta q_\theta{}^T N_\theta N_\theta{}^T \ddot{q}_\theta]_{x=x_S} + GJ\int_0^L \delta q_\theta{}^T N'_\theta N'^T_\theta q_\theta dx = 0$$

Here, the mass matrix, stiffness matrix, and generalized coordinate vector of the above equations are defined as follows:

$$[M] = \begin{bmatrix} \int_0^L mN_w N_w{}^T dx + M_s[N_w N_w{}^T]_{x=x_S} & \int_0^L meN_w N_\theta{}^T dx \\ \int_0^L meN_\theta N_w{}^T dx & \int_0^L mk_m{}^2 N_\theta N_\theta{}^T dx + I_s[N_\theta N_\theta{}^T]_{x=x_S} \end{bmatrix} \quad (5)$$

$$[K] = \begin{bmatrix} EI\int_0^L N''_w N''^T_w dx & 0 \\ 0 & GJ\int_0^L N'_\theta N'^T_\theta dx \end{bmatrix}$$

$$\{q\} = \begin{Bmatrix} q_w \\ q_\theta \end{Bmatrix}$$

Assuming a simple harmonic motion, we can substitute $q(t) = \bar{q}e^{i\omega t}$ into the equations of motion such that the generalized coordinates are given in the form $q_w(t) = \{\bar{q}_w\}e^{i\omega t}$ , $q_\theta(t) = \{\bar{q}_\theta\}e^{i\omega t}$ . Solving the governing equation of the system, $[M]\{\ddot{q}\} + [K]\{q\} = 0$ gives the eigenfrequencies of the beam with store masses in the coupled bending-torsional motion.

## III. Results and Discussion

This section presents the dynamic analysis results of the cantilever beam carrying external stores. We mainly carry out a free vibrational analysis for two different cases in order to show the effect of the external store on the dynamical behavior of the beams. The first case of the results is presented for the beam carrying one mass, whereas the second case is presented for the beam carrying two separate masses. In both cases, we model the beam with a kite-type cross section. In addition, the results in both cases are obtained using the EGM and compared using the FEM. The results of the FEM are handled by performing a modal analysis in ANSYS® software. Lastly, all the frequencies are both tabulated in tables and plotted in figures, and the correlation between the EGM and FEM is shown.

Before proceeding to the results section, we will first validate the accuracy of our dynamic model by considering a special case from the open literature [1–3]. A beam with a square cross section carrying a tip mass is chosen, and the obtained results are compared in Table 1. As the center of gravity coincides with the shear center [take $e = 0$ in Eq. (1)] for a square cross section, the eigenfrequencies of this beam are decoupled in bending and torsional modes. Table 1 reveals that our results, which are provided by the EGM, are in excellent agreement with [1–3] for varying values of tip mass. As discussed in a previous study by Wang [13], we similarly observe that the eigenfrequencies of the beam with a tip mass decrease as the mass increases.

TABLE I. Eigenfrequencies for an Uncoupled Beam with a Point Mass at the Tip.

| | $\omega_1$ (rad/s) | | | | $\omega_2$ (rad/s) | | | | $\omega_3$ (rad/s) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **$M_S$** | **Present** | **[3]** | **[1, 2]** | **Error%** | **Present** | **[3]** | **[1, 2]** | **Error%** | **Present** | **[1, 2]** | **Error%** |
| **0.2** | 2.61 | 2.61 | 2.61 | 0 | 18.21 | 18.38 | 18.21 | 0 | 53.56 | 53.56 | 0 |
| **1** | 1.56 | 1.56 | 1.56 | 0 | 16.25 | 16.56 | 16.25 | 0 | 50.9 | 50.9 | 0 |
| **2** | 1.16 | 1.16 | 1.16 | 0 | 15.87 | 16.19 | 15.86 | 0.06 | 50.45 | 50.45 | 0 |
| **5** | 0.76 | 0.76 | 0.76 | 0 | 15.6 | 15.95 | 15.6 | 0 | 50.17 | 50.16 | 0.02 |
| **10** | 0.54 | 0.54 | 0.54 | 0 | 15.51 | 15.87 | 15.51 | 0 | 50.07 | 50.06 | 0.02 |
| **100** | 0.17 | 0.17 | 0.17 | 0 | 15.43 | 15.79 | 15.43 | 0 | 49.98 | 49.97 | 0.02 |

TABLE II. Coupled Bending-torsional Beam and Mass Properties.

| | | | |
|---|---|---|---|
| Area | *A* | 26 | $m^2$ |
| Mass per unit length | *m* | 7.02E − 05 | kg/m |
| Poisson's ratio | *v* | 0.3 | — |
| Young's modulus | *E* | 7.00E + 04 | $N/m^2$ |
| Shear center (*y*, *z*) | *SC* | (1.468, 0) | m |
| Centroid (*y*, *z*) | *CG* | (2.256, 0) | m |
| External mass | *M* | 0.01 | kg |
| Length of beam | *l* | 200 | m |

TABLE III. EIGENFREQUENCIES FOR A COUPLED BENDING-TORSIONAL BEAM WITH A MASS.

| | $\xi = L/8$ | | | $\xi = L/4$ | | | $\xi = L/2$ | | | $\xi = 3L/4$ | | | $\xi = L$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Mode** | **EGM** | **FEM** | **Error%** | **EGM** | **FEM** | **Error%** | **EGM** | **FEM** | **Error%** | **EGM** | **FEM** | **Error%** | **EGM** | **FEM** | **Error%** |
| $\omega_1$ | 1.93 | 1.93 | 0.02 | 1.91 | 1.91 | 0.02 | 1.67 | 1.67 | 0.02 | 1.29 | 1.29 | 0.02 | 0.98 | 0.98 | 0.01 |
| $\omega_2$ | 5.80 | 5.80 | 0.13 | 5.73 | 5.72 | 0.14 | 5.02 | 5.01 | 0.17 | 3.89 | 3.88 | 0.14 | 2.94 | 2.93 | 0.09 |
| $\omega_3$ | 11.77 | 11.75 | 0.17 | 9.66 | 9.64 | 0.20 | 8.36 | 8.35 | 0.15 | 11.97 | 11.96 | 0.13 | 9.09 | 9.08 | 0.10 |
| $\omega_4$ | 28.46 | 28.26 | 0.69 | 23.86 | 23.75 | 0.44 | 25.12 | 24.88 | 0.93 | 27.87 | 27.81 | 0.22 | 27.19 | 27.10 | 0.36 |
| $\omega_5$ | 35.30 | 34.92 | 1.08 | 28.99 | 28.59 | 1.39 | 33.91 | 33.81 | 0.31 | 35.91 | 35.59 | 0.91 | 28.16 | 28.09 | 0.26 |
| $\omega_6$ | 49.98 | 49.47 | 1.03 | 56.81 | 56.44 | 0.65 | 53.29 | 52.78 | 0.94 | 57.75 | 57.58 | 0.30 | 57.76 | 57.48 | 0.48 |
| $\omega_7$ | 70.85 | 70.85 | 0.00 | 69.19 | 69.83 | −0.92 | 70.82 | 70.81 | 0.01 | 70.40 | 70.84 | −0.63 | 70.83 | 70.82 | 0.02 |

TABLE IV. EIGENFREQUENCIES FOR AN UNCOUPLED BENDING-TORSIONAL BEAM WITH TWO MASSES.

| | $\xi = L/8$ | | | $\xi = L/4$ | | | $\xi = L/2$ | | | $\xi = 3L/4$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Mode** | **EGM** | **FEM** | **Error%** | **EGM** | **FEM** | **Error%** | **EGM** | **FEM** | **Error%** | **EGM** | **FEM** | **Error%** |
| $\omega_1$ | 0.35 | 0.35 | 0.01 | 0.35 | 0.35 | 0.01 | 0.35 | 0.35 | 0.01 | 0.34 | 0.34 | 0.01 |
| $\omega_2$ | 1.06 | 1.05 | 0.08 | 1.05 | 1.05 | 0.07 | 1.05 | 1.05 | 0.08 | 1.04 | 1.03 | 0.08 |
| $\omega_3$ | 8.42 | 8.41 | 0.08 | 7.36 | 7.35 | 0.14 | 5.41 | 5.40 | 0.12 | 5.95 | 5.95 | 0.10 |
| $\omega_4$ | 24.23 | 24.11 | 0.48 | 18.95 | 18.88 | 0.36 | 16.23 | 16.12 | 0.71 | 17.87 | 17.76 | 0.60 |
| $\omega_5$ | 25.19 | 25.09 | 0.40 | 22.07 | 21.90 | 0.78 | 26.24 | 26.21 | 0.14 | 21.55 | 21.48 | 0.29 |

### A. *Coupled Bending-Torsional Beam Carrying a Mass*

In this first set, we demonstrate the results of the dynamic analysis of a cantilever beam carrying one mass. By altering the span location of the mass, we aim to investigate the effect of the location of the external mass on the frequencies of the beam. The material and geometric properties of the beam used in this section are listed in Table II.

We consider the beam with an external mass, $M_S$, located at a distance $\xi$ from the fixed end. The cross section of the beam is chosen as a kite-type section, whose sketch is given in Fig. 3.

The condition of $\xi = 0$ refers to the fixed-free beam natural frequency itself. The eigenfrequency results displayed in Fig. 4 and tabulated in Table 3 are compared with the results of the finite element analysis using ANSYS® software. The beam structure is divided into 200 elements using linear, six-degree-of-freedom BEAM188 elements. The results are plotted up to the seventh natural frequency where the torsional mode is observed. The results of both methods overlap with each other, even for higher-order modes. The bending frequencies decrease as the mass gets closer to the tip, which can be clearly observed at the first two modes. Higher-order modes do not show a stable trend when compared to the lower-order modes.

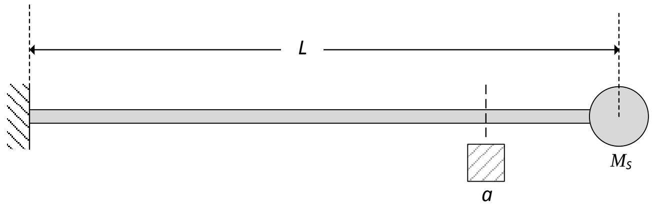


Figure 2. Uncoupled bending-torsional beam with a point mass.

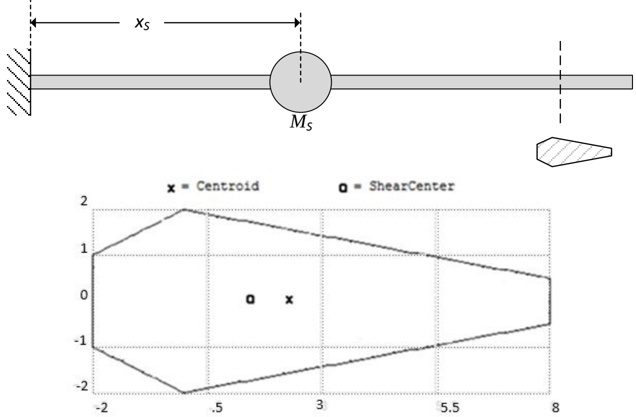


Figure 3. Coupled bending-torsional beam.

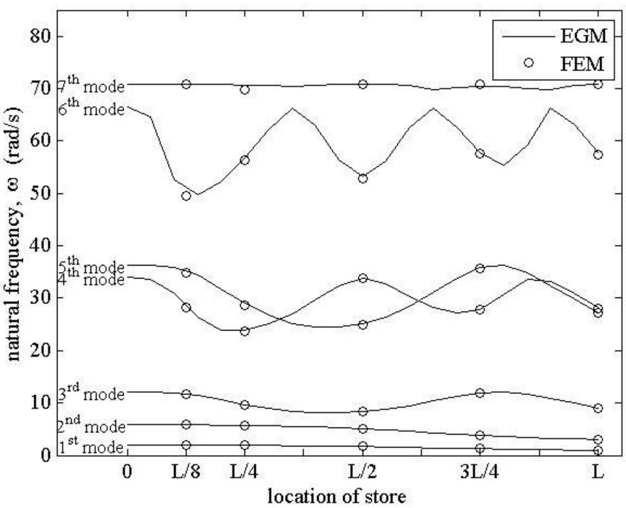


Figure 4. Natural frequencies of a coupled bending-torsional beam with a mass (refer to Fig. 3 for configuration).

## B. *Coupled Bending-Torsional Beam with Multiple Masses*

In this section, we present the results of the dynamical analysis of a coupled bending-torsional beam carrying two masses. The material properties of the beam are identical to those of the previous numerical example, which is defined in Table II. The sketch of the cross section of the beam and the beam geometry are given in Figs. 3(b) and 5, respectively. The external mass, $M_{S1}$, is chosen as stationary at the tip of the beam, whereas the point mass, $M_{S2}$, is chosen to be placed in various locations along the beam span.

An analysis similar to that in the previous case is conducted, and the obtained results are compared with the ones of the FEM and given in Table IV. As seen from Table IV and Fig. 6, the natural frequency results are provided with subtle changes as the location of the point mass is altered. As expected, such behavior vanishes at higher-order modes. Additionally, including a second mass resulted in a decrease in the frequencies, regardless of the location of the second mass.

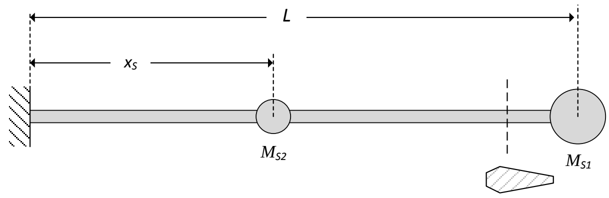


Figure 5. Coupled bending-torsional beam with two masses.

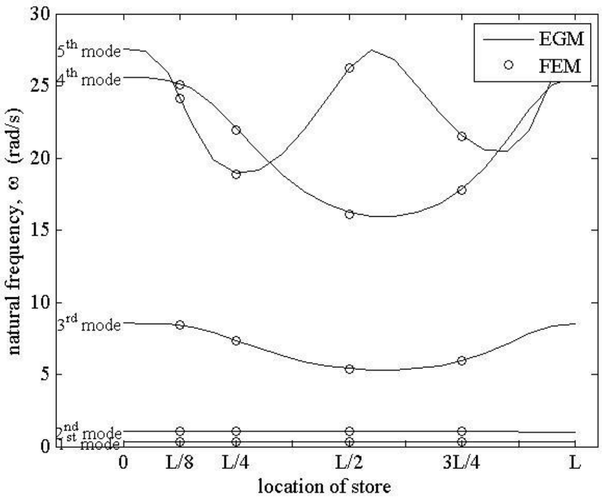


Figure 6. Natural frequencies of a coupled bending-torsional beam with two masses.

## IV. CONCLUSIONS

In this study, we focused on the dynamical analysis of the coupled bending-torsional behavior of a cantilever beam carrying external masses. The eigenfrequencies were found using both EGM and the finite element software ANSYS®. The obtained results of both methods were compared with each other. We found that the EGM provides a high accuracy with rapid convergence and maintains this ability for even higher-order modes.

Two different beam-mass configurations were presented and analyzed in order to demonstrate the effects of external masses on the dynamical characteristics of the beam. We initially validated our results using EGM for a beam with a square cross section with a tip mass against the open literature, showing a very good agreement. In the first beam-mass configuration, we placed a mass at different span locations and performed free vibrational analyses to obtain the natural frequencies of the beam. The results showed that as the mass gets closer to the tip, a larger decrease in the natural frequency is observed. In the second configuration, the dynamical results of a beam carrying two masses were presented. For this one, the first mass was kept stationary at the tip of the beam and the second mass was placed at various locations along the span. The results of this case similarly showed that the tip mass provides a more stable dynamical behavior, regardless of the different placements of the second mass. In addition, we observed that higher-order modes do not show a notable trend and we see that natural frequencies vary as the mass take different locations along the span.

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